Chapter 5

Ionospheric currents

Agenda:

- Based on the momentum equation for ions and electrons we are going to derive expressions for the Hall- and the Petersen conductivities.
- We are going to study how the current density vector $\vec{j}$ rotates with height.
- We are going to discuss the physical limits for electric conductivity at the bottom side *E*-layer and topside within the *F*-layer

The simplest expression for current density is given as:

$$\vec{j} = n_e e \left(\vec{v}_i - \vec{v}_e\right) \qquad \left[\frac{1}{m^3} C \frac{m}{s} = \frac{A}{m^2}\right] \quad (1)$$

$\vec{j}$ is proportional with the electron density n_e, and the difference in ion and electron velocity. In this lecture we are going to establish the following equation for the current density

$$\vec{j} = \vec{\sigma} \cdot \vec{E} \qquad (2)$$

Momentum equation for ions and electrons

$$n_i m_i \frac{d\vec{v}_i}{dt} = N_i e \left(\vec{E} + \vec{v}_i \times \vec{B}\right) + n_i m g - \nabla \rho_i - n_i m_i \gamma_i \left(\vec{v}_i - \vec{v}_n\right) \quad (3)$$

$$n_e m_e \frac{d\vec{v}_e}{dt} = -N_e e \left(\vec{E} + \vec{v}_i \times \vec{B}\right) + n_e m g - \nabla \rho_e - n_e m_e \gamma_e \left(\vec{v}_e - \vec{v}_n\right) \quad (4)$$

For the *E*-region and *F*-region of the ionosphere we can neglect:

- Pressure force
- Gravity force
- Acceleration term (*)

(*) Argument

Simple model for the collision frequency:

$\gamma_{in} = 2.6 \cdot 10^{-15} < M >^{-1/2} \quad \{N_n\}$

$\gamma_{en} = 5.4 \cdot 10^{-16} T^{1/2} \qquad \{N_n\}$

$\{n_n\} = 5 \cdot 10^{17} m^{-3}$ Neutral density in the E-region.

$T = 250K$

$M = 29u$ Mixture of N_2, O and NO

=> $\gamma_{in} = 250 s^{-1}$ and $\gamma_{en} \cong 4300 s^{-1}$

Assume that the acceleration term can be neglected:

$$\left| m_i \frac{d\vec{v}_i}{dt} \right| << \left| m_i \gamma_i (\vec{v}_i - \vec{v}_n) \right|$$

$$\frac{v_i}{T_i} << \gamma_{in} v_i \text{ (Neglect the neutral wind)}$$

$$T_i >> \frac{1}{\gamma_{in}} = 4 \cdot 10^{-3} s$$

$$T_e >> \frac{1}{\gamma_{en}} = 2.3 \cdot 10^{-4} s$$

Normally we are not interested in time-scales less than a second, and the acceleration term is negligible.

Simplified momentum equation

$$\vec{v}_i = \frac{e}{m_i \gamma_{in}} \left(\vec{E} + \vec{v}_i \times \vec{B} \right) + \vec{v}_n \qquad (5)$$

$$\vec{v}_e = -\frac{e}{m_e \gamma_{en}} \left(\vec{E} + \vec{v}_e \times \vec{B} \right) + \vec{v}_n \qquad (6)$$

In order to single $\vec{v}_i$ and $\vec{v}_e$ we need to find $\vec{v}_{i,e} \times \vec{B}$ and $\vec{v}_{i,e} \cdot \vec{B}$

$$\vec{v}_i \times \vec{B} = \frac{e}{m_i \gamma_{in}} \left\{ \vec{E} \times \vec{B} + \left(\vec{v}_i \times \vec{B} \right) \times \vec{B} \right\} + \vec{v}_n \times \vec{B}$$

$$\vec{v}_i \times \vec{B} = \frac{e}{m_i \gamma_{in}} \left\{ \vec{E} \times \vec{B} + \left(\vec{v}_i \times \vec{B} \right) \vec{B} - B^2 \vec{v}_i \right\} + \vec{v}_n \times \vec{B} \qquad (7)$$

$$\vec{v}_i \cdot \vec{B} = \frac{e}{m_i \gamma_{in}} \left\{ \vec{E} \cdot \vec{B} + \left(\vec{v}_i \times \vec{B} \right) \cdot \vec{B} \right\} + \vec{v}_n \cdot \vec{B}$$

$$\vec{v}_i \cdot \vec{B} = \frac{e}{m_i \gamma_{in}} \vec{E} \cdot \vec{B} + \vec{v}_n \cdot \vec{B} \qquad (8)$$

Insert equation 8 in equation 7:

$$\vec{v}_i \times \vec{B} = \frac{e}{m_i\ \gamma_{in}} \left\{ \vec{E} \times \vec{B} + \left(\frac{e}{m_i \gamma_{in}} \vec{E} \cdot \vec{B} + \vec{u} \cdot \vec{B} \right) \vec{B} - B^2 \vec{v}_i \right\} + \vec{v}_n \times \vec{B} \qquad (9)$$

Then insert equation 9 in equation 5:

$$\vec{v}_i = \frac{e}{m_i\ \gamma_{in}} \left\{ \vec{E} + \frac{e}{m_i \gamma_{in}} \left(\vec{E} \times \vec{B} \left(\frac{e}{m_i \gamma_{in}} \vec{E} \cdot \vec{B} + \vec{v}_n \cdot \vec{B} \right) \vec{B} - B^2 \vec{v}_i \right) + \vec{v}_n \times \vec{B} \right\} + \vec{v}_n$$

Isolate $\vec{v}_i$ on the left hand side:

$$\left(1 + \frac{e^2 B^2}{m_i^2 \gamma_{in}^2} \right) \vec{v}_i = \frac{e}{m_i\ \gamma_{in}}\ \vec{E} + \frac{e^2}{m_i^2\ \gamma_{in}^2}\ \vec{E} \times \vec{B} + \frac{e^3}{m_i^3\ \gamma_{in}^3}\ \left(\vec{E} \cdot \vec{B} \right) \vec{B} - \frac{e^2}{m_i^2\ \gamma_{in}^2}\ \left(\vec{v}_n \cdot \vec{B} \right) \vec{B} + \frac{e}{m_i\ \gamma_{in}}\ \vec{v}_n \times \vec{B} + \vec{v}_n$$

Introduce the ion gyro-frequency given by

$$\omega_i = \frac{eB}{m_i}$$

$$\left(1 + \left(\frac{\omega_i}{\gamma_{in}} \right)^2 \right) \vec{v}_i = \frac{\omega_i}{\gamma_{in}}\ \frac{E}{B} + \frac{\omega_i^2}{\gamma_{in}^2} \frac{\vec{E} \times \vec{B}}{B^2} + \frac{\omega_i^3}{\gamma_{in}^3} \left(\frac{\left(\vec{E} \cdot \vec{B} \right) \vec{B}}{B^3} \right) + \frac{\omega_i^2}{\gamma_{in}^2} \frac{\left(\vec{v}_n \cdot \vec{B} \right)}{B^2} + \frac{\omega_i}{\gamma_{in}} \frac{\vec{v}_n \times \vec{B}}{B} + \vec{v}_n$$

$$\vec{v}_i = \frac{\gamma_{in}^2}{\omega_i^2 + \gamma_{in}^2} \left[\frac{\omega_i}{\gamma_{in} B} \left(\vec{E} + \vec{v}_n \times \vec{B} \right) + \frac{\omega_i^2}{\gamma_{in}^2 B^2} \left(\vec{E} \times \vec{B} + \left(\vec{v}_n \cdot \vec{B} \right) \vec{B} \right) + \frac{\omega_i^3}{\gamma_{in}^3 B^3}\ (\vec{E} \cdot \vec{B})\ \vec{B} + \vec{u} \right]$$

$$\vec{v}_i = \frac{\omega_i\ \gamma_{in}}{\omega_i^2 + \gamma_{in}^2} \left(\frac{\vec{E} + \vec{v}_n \times \vec{B}}{B} \right) + \frac{\omega_i^2}{\omega_i^2 + \gamma_{in}^2} \frac{\left(\vec{E} \times \vec{B} + \left(\vec{v}_n \cdot \vec{B} \right) \vec{B} \right)}{B^2} + \frac{\omega_i^3}{(\omega_i^2 + \gamma_{in}^2) \gamma_{in}} \frac{(\vec{E} \cdot \vec{B})\ \vec{B}}{B^3} + \frac{\gamma_{in}^2}{\omega_i^2 + \gamma_{in}^2} \vec{v}_n$$

Let us no decompose the electric field in one component along the magnetic field and one component perpendicular to the magnetic field:

$$\begin{aligned}
\vec{E} &= E_{\parallel} + E_{\perp} \\
\left(\vec{E} + \vec{v}_n \times \vec{B} \right) \times \vec{B} &= \vec{E}_{\perp} \times \vec{B} + \left(\vec{v}_n \cdot \vec{B} \right) \vec{B} - \vec{v}_n B^2 \\
\vec{E} \times \vec{B} + \left(\vec{v}_n \cdot \vec{B} \right) \vec{B} &= \left(\vec{E} + \vec{v}_n \times \vec{B} \right) \times \vec{B} + \vec{v}_n B^2 = \left(\vec{E}_{\perp} + \vec{v}_n \times \vec{B} \right) \times \vec{B} \\
&= \left(E_{\perp} + \vec{v}_n \times \vec{B} \right) \times \vec{B} \\
\left(\vec{E} \cdot \vec{B} \right) \vec{B} &= \vec{E}_{\parallel} B^2
\end{aligned}$$

$$\vec{v}_i = \frac{\omega_i \gamma_{in}}{\omega_i^2 + \gamma_{in}^2} \frac{\left(\vec{E}_\perp + \vec{v}_n \times \vec{B}\right)}{B} + \frac{\omega_i^2}{\omega_i^2 + \gamma_{in}^2} \frac{\left(\vec{E}_\perp + \vec{v}_n \times \vec{B}\right) \times \vec{B}}{B^2} + \frac{\omega_i \gamma_{in}}{\omega_i^2 + \gamma_{in}^2} \frac{\vec{E}_{pa}}{B} + \frac{\omega_i^3}{\left(\omega_i^2 + \gamma_{in}^2\right)\gamma_{in}} \frac{\vec{E}_\parallel}{B} +$$

$$\frac{\omega_i^2}{\omega_i^2 + \gamma_{in}^2} \vec{v}_n + \frac{v_{in}^2}{\omega_i^2 + \gamma_{in}^2} \vec{v}_n$$

$$\vec{v}_i - \vec{v}_n = \frac{\omega_i \gamma_{in}}{\omega_i^2 + \gamma_{in}^2} \frac{\vec{E}'_\perp}{B} + \frac{\omega_i^2}{\omega_i^2 + \gamma_{in}^2} \frac{\vec{E}'_\perp \times \vec{B}}{B^2} + \frac{\omega_i^2}{\gamma_{in}^2} \frac{\vec{E}_\parallel}{B}$$

where

$$\vec{E}'_\perp = \vec{E}_\perp + \vec{v}_n \times \vec{B}$$

We have now obtained the following expressions for the electron and ion velocity:

$$\vec{v}_i = \vec{v}_n + \frac{\omega_i \gamma_{in}}{\omega_i^2 + \gamma_{in}^2} \frac{\vec{E}'_\perp}{B} + \frac{\omega_i^2}{\omega_i^2 + \gamma_{in}^2} \frac{\vec{E}'_\perp \times \vec{B}}{B^2} + \frac{\omega_i}{\gamma_{in}} \frac{\vec{E}_\parallel}{B} \quad (10)$$

$$\vec{v}_e = \vec{v}_n - \frac{\omega_e \gamma_{en}}{\omega_e^2 + \gamma_{en}^2} \frac{\vec{E}'_\perp}{B} + \frac{\omega_e^2}{\omega_e^2 + \gamma_{en}^2} \frac{\vec{E}'_\perp \times B}{B^2} - \frac{\omega_e}{\gamma_{en}} \frac{\vec{E}_\parallel}{B} \quad (11)$$

Inserting Eq. 10 and 11 into Eq. 1 we get

$$\vec{j} = n_e e \left\{ \left(\frac{\omega_e \gamma_{en}}{\omega_e^2 + \gamma_{en}^2} + \frac{\omega_i \gamma_{in}}{\omega_i^2 + \gamma_{in}^2} \right) \frac{\vec{E}'_\perp}{B} + \left(\frac{\omega_e^2}{\omega_e^2 + \gamma_{en}^2} - \frac{\omega_i^2}{\omega_i^2 + \gamma_{in}^2} \right) \frac{\vec{B} \times \vec{E}'_\perp}{B^2} + \left(\frac{\omega_e}{\gamma_{en}} + \frac{\omega_i}{v_{in}} \right) \frac{\vec{E}_\parallel}{B} \right\}$$

which can be written on the form

$$\vec{j} = \sigma_p \vec{E}'_\perp + \sigma_H \frac{\vec{B} \times \vec{E}'_\perp}{B} + \sigma_\parallel \vec{E}_\parallel \quad (12)$$

where

$$\sigma_P = \frac{e}{B} n_e \left\{ \frac{\omega_e \gamma_{en}}{\omega_e^2 + \gamma_{en}^2} + \frac{\omega_i \gamma_i}{\omega_i^2 + \gamma_{in}^2} \right\} = \frac{e}{B} n_e \left(\alpha_e + \alpha_i\right) \quad (13)$$

$$\sigma_H = \frac{e}{B} n_e \left\{ \frac{\omega_e^2}{\omega_e^2 + \gamma_{en}^2} - \frac{\omega_i^2}{\omega_i^2 + \gamma_{in}^2} \right\} = \frac{e}{B} n_e \left(\beta_e - \beta_i\right) \quad (14)$$

σ_P and σ_H are the Pedersen and Hall conductivities, respectively. The Pedersen conductivity is associated with the Pedersen current along the electric field but perpendicular to the magnetic field. The Hall conductivity is associated with the Hall current perpendicular to the magnetic and the electric field. The $\alpha_{e,i}$ and $\beta_{e,i}$ on the right are the Pedersen and Hall mobility coefficients for electrons and ions:

$$\alpha_e = \frac{\omega_e \gamma_{en}}{\omega_e^2 + \gamma_{en}^2}$$

$$\alpha_i = \frac{\omega_i \gamma_{in}}{\omega_i^2 + \gamma_{in}^2}$$

$$\beta_e = \frac{\omega_e^2}{\omega_e^2 + \gamma_{en}^2}$$

$$\beta_i = \frac{\omega_i^2}{\omega_i^2 + \gamma_{in}^2}$$

Figure 5.2 shows the altitude variation of these coefficients. By inspection of figure 5.2 we can easily quantify the Hall and Pedersen mobility coefficients, $k_H = \beta_e - \beta_i$ and $k_P = \alpha_e + \alpha_i$, at three different altitudes:

	α_e	α_i	β_e	β_i	$k_P = \alpha_e + \alpha_i$	$k_H = \beta_e - \beta_i$
90km	0	0	1	0	0	1
125km	0	0.5	1	0.5	0.5	0.5
180km	0	0	1	1	0	0

Table 5.1

Let us now consider how do $\vec{v}_i$ and $\vec{v}_e$ rotates with height.
Assume that $E_{||} = 0$, $v_n = 0$.
Equations (10) and (11) then become

$$\vec{v}_i = \frac{\omega_i \gamma_{in}}{\omega_i^2 + \gamma_{in}^2} \frac{\vec{E}_\perp}{B} + \frac{\omega_i^2}{\omega_i^2 + \gamma_{in}^2} \frac{\vec{E}_\perp \times \vec{B}}{B^2}$$

$$\vec{v}_e = -\frac{\omega_e \gamma_{en}}{\omega_e^2 + \gamma_{en}^2} \frac{\vec{E}_\perp}{B} + \frac{\omega_e^2}{\omega_e^2 + \gamma_{en}^2} \frac{\vec{E}_\perp \times \vec{B}}{B^2}$$

$$\tan\theta_i = \frac{\omega_i}{\gamma_{in}} \qquad (15)$$

$$\tan\theta_e = \frac{\omega_e}{\gamma_{en}} \qquad (16)$$

Figure 5.1 shows the electron-neutral and ion-neutral collision frequency and the ion and electron gyro frequency versus height. By reading out values from this figure 5.1 we easily find the angle θ_e and θ_i for three different altitudes as shown in Table 5.2 below.

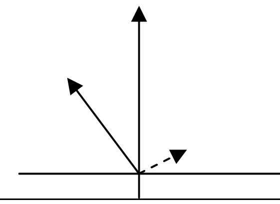
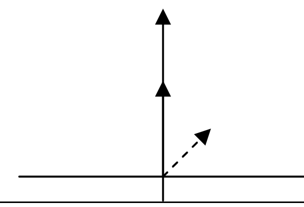
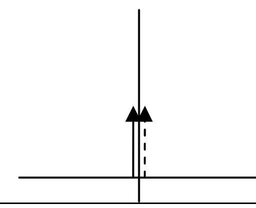
80 km	$\gamma_{en} = \omega_e$ $=>\theta_e$=45° $\gamma_{en} >> \omega_i$ $=>\theta_i$=0°	
125 km	$\gamma_{en} << \omega_e$ => θ_e=90° $\gamma_{en} = \omega_e$ => θ_i=45°	
200 km	$\gamma_{en} << \omega_e$ => θ_e=90° $\gamma_{en} << \omega_i$ => θ_e=90°	

Table 5.2: The ion vector rotates anti-clockwise and is indicated by a dashed arrow. The electron vector is fill line arrow rotating clock-wise.

Please notice that $\vec{v}_i$ and $\vec{v}_e$ are in the same direction and have the same magnitude above 200 km. Therefore now current according to Eq. 1. Also according to Table 5.1, the electrons and ions have no mobility across the magnetic field at this altitude. At 80 km there will be practically now current because the electron density is very low.

Let us now organize Eq. 12 on a matrix form

$$\begin{pmatrix} j_1 \\ j_2 \\ j_3 \end{pmatrix} = \begin{pmatrix} \sigma_P & 0 & 0 \\ 0 & \sigma_H & 0 \\ 0 & 0 & \sigma_{pa} \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}$$

where $\vec{E}_1 = \vec{E}_\perp$, $E_2 = \frac{\vec{B} \times \vec{E}_\perp}{B}$, $\vec{E}_3 = \vec{E}_\parallel$

However, this is not a convenient co-ordinate system as the direction of the E-field is a subject to rapid variations. Let us therefore introduce a co-ordinate system where x is magnetic north, y is negative east and z is always the magnetic field.

$$\begin{aligned} \vec{B} &= B\hat{z} \\ \vec{E} &= E_x\hat{x} + E_y\hat{y} + E_z\hat{z} \\ \vec{E}_\perp &= E_x\hat{x} + E_y\hat{y} \end{aligned}$$

Inserted in equation 12

$$\left(\vec{j} = \sigma_P \, (E_x\hat{x} + E_y\hat{y}) \; + \sigma_H \; \frac{B\hat{z} + (E_x\hat{x} + E_y\hat{y})}{B} \; + \; \sigma_\parallel E_z\hat{z} \; = \; (\sigma_P E_x - \sigma_H E_y)\hat{x} \; + \; (\sigma_p E_y + \sigma_H E_x)\hat{y} \; + \; \sigma_\parallel \right) E_z\hat{z}$$

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \begin{pmatrix} \sigma_P & -\sigma_H & 0 \\ \sigma_H & \sigma_P & 0 \\ 0 & 0 & \sigma_\parallel \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

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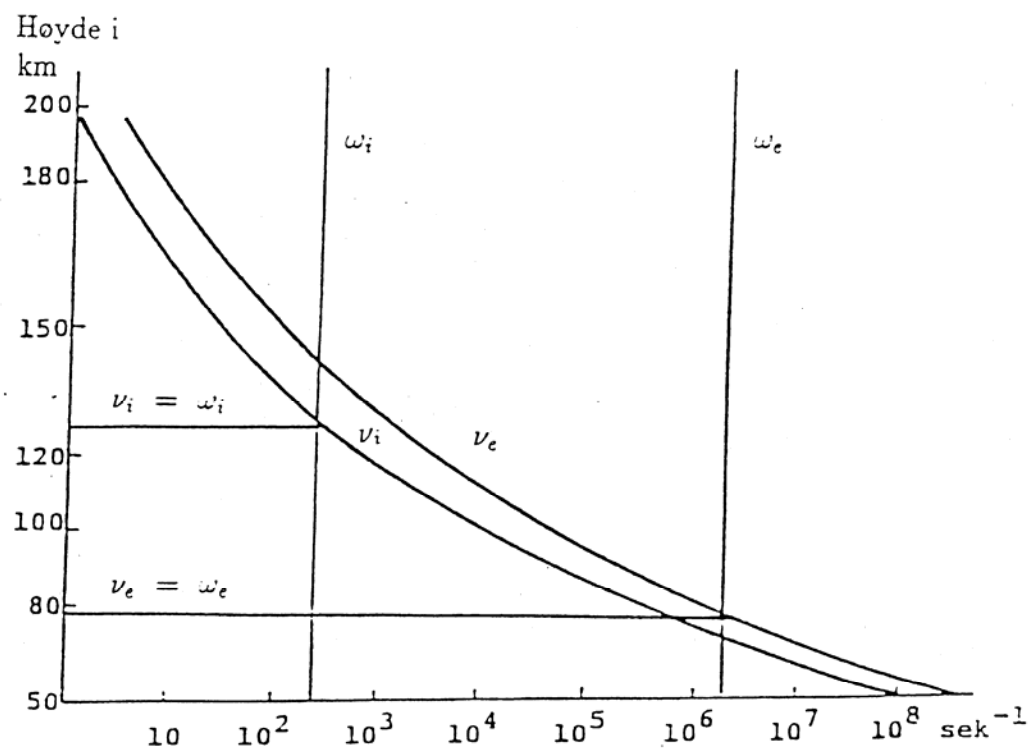

Figure 5.1

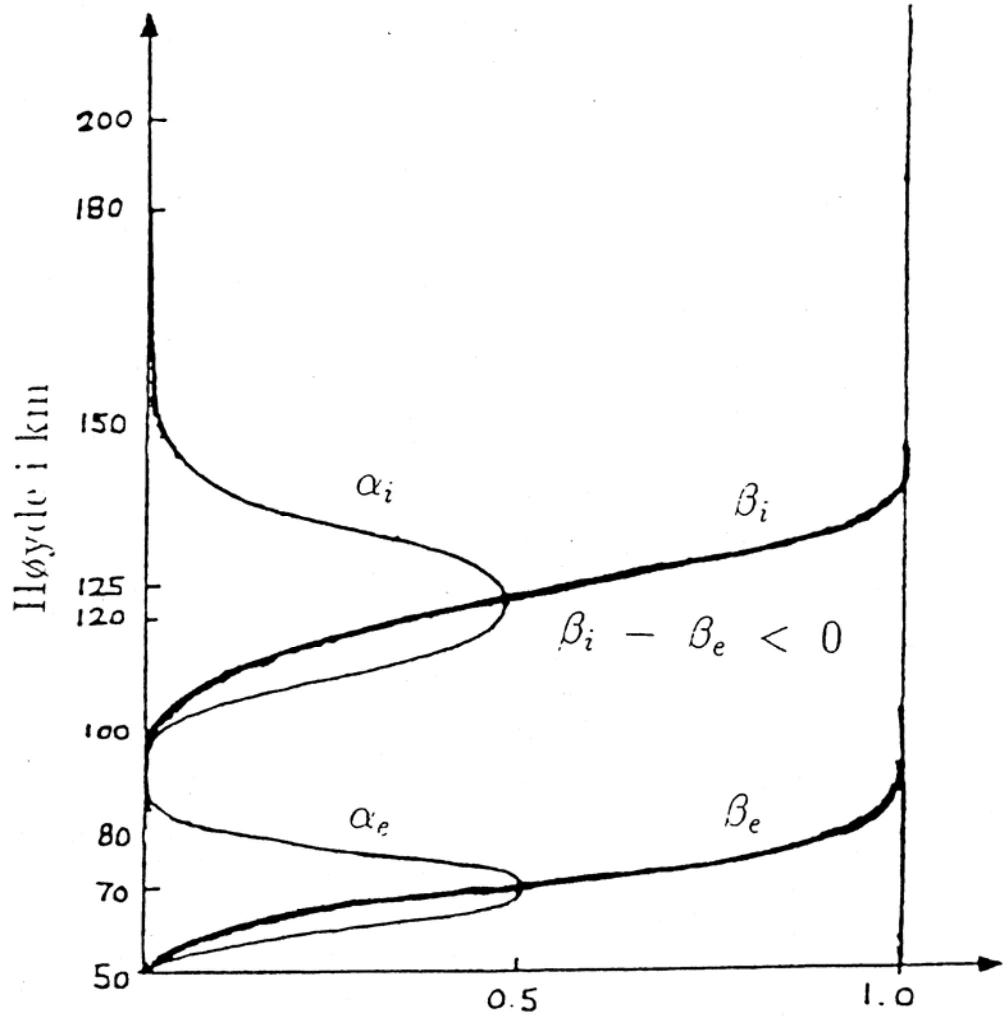
Høyde i km
200
180
150
125
120
100
80
70
50
0.5
1.0
α_i
β_i
$\beta_i - \beta_e < 0$
α_e
β_e

Figure 5.2